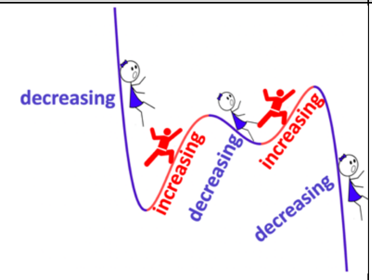
First Derivative $\frac{dy}{dx}$ (talks about slope)		Second Derivative $\frac{d^2y}{dx^2}$ (talks about concavity)	
Increasing and decreasing	Stationary/Turning Points (Max or Min)	Concave Up (convex) Concave Down (aka concave)	Points of Inflection

Increasing and decreasing



Increasing:
This is where the slope is positive which means greater than zero. Imagine someone climbing up a hill.

solve $\frac{dy}{dx} > 0$

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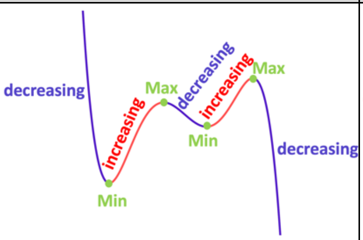
Decreasing:
This is where the slope is negative which means less than zero. Imagine someone sliding down a hill.

solve $\frac{dy}{dx} < 0$

Important: Remember that when we solve an inequality that is a quadratic or higher we must use the sign change test or graph! We cannot just guess the signs!

Stationary/Turning Points (Max or Min)



Max/Min (Turning/Stationary Points):
This is where increasing changes to decreasing or vice versa. The slope is neither positive, nor negative, it is zero!

solve $\frac{dy}{dx} = 0$

To verify whether a max or min:

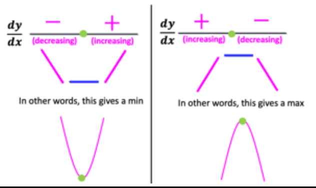
Way 1: Plug into $\frac{d^2y}{dx^2}$

Plug the x value found into $\frac{d^2y}{dx^2}$ and if

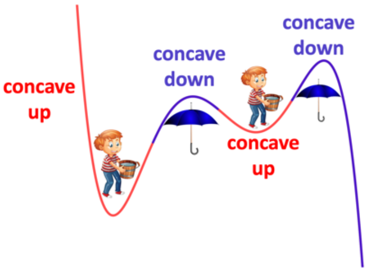
$\frac{d^2y}{dx^2} > 0 \Longrightarrow$ min
$\frac{d^2y}{dx^2} < 0 \Longrightarrow$ max

Way 2: Sign change number line test for $\frac{dy}{dx}$

We plug in an x value just below and just above the x value found into $\frac{dy}{dx}$. If $\frac{dy}{dx}$ changes sign from negative $(-)$ to $(+)$ then min and if $\frac{dy}{dx}$ changes from positive $(+)$ to negative $(-)$ then max.



Concave Up (convex) / Concave Down (aka concave)



Concave Up/Convex:
This is where the concavity is positive. Imagine a bowl or the inside of a bucket. Concave up means the rainwater would be held by the bowl/bucket. The curve acts in the same way, hence if we poured water on the curve, it would be held by the curve.

Solve $\frac{d^2y}{dx^2} > 0$

Concave Down/Concave:
This is where the concavity is negative. Imagine an upside-down bowl or an umbrella. Concave down means that the rainwater would spill, so if we poured water on the curve, it would roll off the curve.

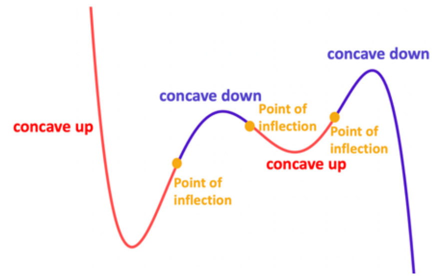
solve $\frac{d^2y}{dx^2} < 0$

Important: Remember that when we solve an inequality that is a quadratic or higher we should use the sign change test or graph! We cannot just guess the signs!

Note: It should now make sense why we have the criteria

$\frac{d^2y}{dx^2} > 0 \Longrightarrow$ min (holds water)
$\frac{d^2y}{dx^2} < 0 \Longrightarrow$ max (spills water)

Points of Inflection



Points of inflection:
These are where concavity changes from concave up to down or vice versa.
The concavity is neither positive, nor negative, it is zero!

solve $\frac{d^2y}{dx^2} = 0$

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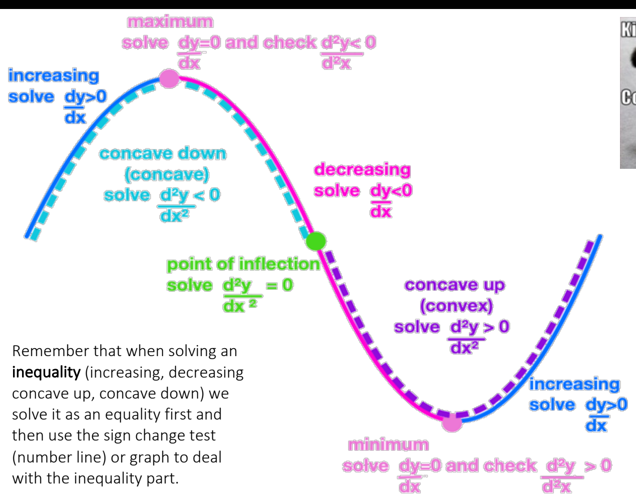
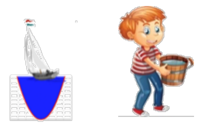
Summary

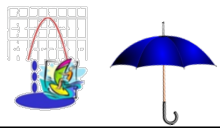
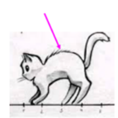


Remember that when solving an **inequality** (increasing, decreasing concave up, concave down) we solve it as an equality first and then use the sign change test (number line) or graph to deal with the inequality part.

Concave Up Looks Like:

holds water	happy cat	yoga - cow pose
		

Concave Down Looks Like:

spills water	scared cat	yoga - cat pose
		

Example to tie all concepts together:

$$y = 3x^5 - 5x^3$$

Find the

i. Max and min points coordinates and state which is which
ii. Regions where the function is increasing
iii. Points on inflection
iv. Regions where the function is concave down

Answer

i.

Maximum/Minimum

Max/min when $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = 15x^4 - 15x^2$$

$$15x^4 - 15x^2 = 0$$

$$x^4 - x^2 = 0$$

We factorise in order to solve (see my solving notes or cheat sheet if you struggle with solving equations such as quadratics and above)

$$x^2(x^2 - 1) = 0$$

Set each bracket equal to 0

$$x^2 = 0, x^2 - 1 = 0$$

$$x^2 = 0, x^2 = 1$$

This gives the solutions

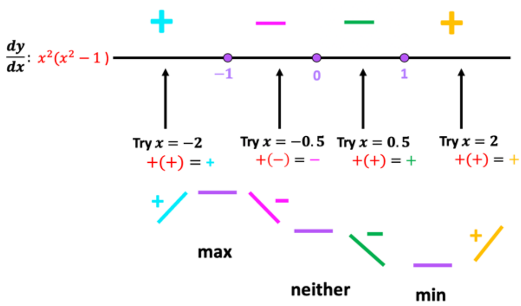
$$x = 0, x = 1, x = -1$$

Let's classify which are min and which are max. There are 2 ways to do this.

Way 1: sign change test on $\frac{dy}{dx}$

Plot all the solutions on a regular number line and plug values in between each set of numbers (each region) into $x^2(x^2 - 1)$

Note the signs of your answers



The signs form the slopes. Check whether the slope lines form a min or max shape

$x = -1$ is a max, $x = 1$ is a min and $x = 0$ is neither

Way 2: Plug x found into $\frac{d^2y}{dx^2}$ (quickest and easiest method)

$$\frac{d^2y}{d^2x} = 60x^3 - 30x$$

Plug $x = 0, x = -1$ and $x = 1$ into $\frac{d^2y}{d^2x} = 60x^3 - 30x$

when $x = 0$:

$\frac{d^2y}{d^2x} = 60(0)^3 - 30(0) = 0$ ∴ neither

when $x = -1$

$\frac{d^2y}{d^2x} = 60(-1)^3 - 30(-1) = -30 < 0$ ∴ max

when $x = 1$:

$\frac{d^2y}{d^2x} = 60(1)^3 - 30(1) = 30 > 0$ ∴ min

We have the x coorindates. Let's find the entire coordinate.

When $x = 0$

$$y = 3(0)^5 - 5(0)^3 = 0$$

When $x = 1$

$$y = 3(1)^5 - 5(1)^3 = -2$$

When $x = -1$

$$y - 3(-1)^5 - 5(-1)^3 = 2$$

$(1, -2)$ min
$(-1, 2)$ max
$(0, 0)$ neither

ii.

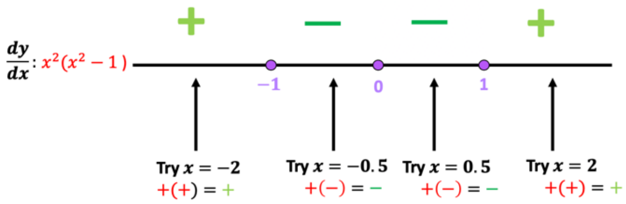
Increasing

Increasing when $\frac{dy}{dx} > 0$

We solve $\frac{dy}{dx} = 0$ and then use a number line to solve the inequality

We already have the points where $\frac{dy}{dx} = 0$ from part i.
We put these on a number line and check the signs

$+$ means increasing
$-$ means decreasing



Increasing when $x < -1$ and $0 < x < 1$

iii.

Points of inflection

Points of inflection when $\frac{d^2y}{dx^2} = 0$

$$\frac{d^2y}{dx^2} = 60x^3 - 30x$$

$$60x^3 - 30x = 0$$

We factorise in order to solve

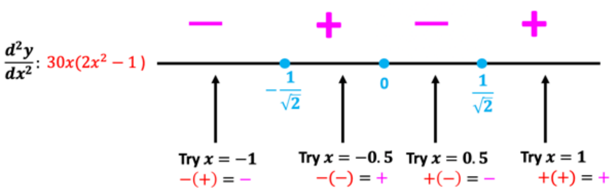
$$30x(2x^2 - 1) = 0$$

Set each bracket equal to 0

$$30x = 0, 2x^2 - 1 = 0$$
$$30x = 0, x^2 = \frac{1}{2}$$
$$x = 0, x = \pm\frac{1}{\sqrt{2}}$$

Let's verify that there are points in inflection

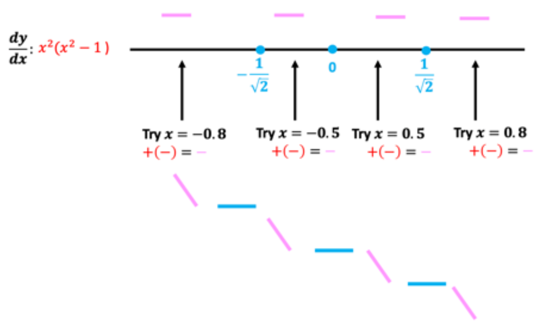
Way 1: Sign change test on $\frac{d^2y}{dx^2}$ (we want a sign change)



All change sign so all are points of inflection

$$x = 0, x = \pm\frac{1}{\sqrt{2}}$$

Way 2: sign change test on $\frac{dy}{dx}$ (we want no sign change)



All do not change sign, so all are points of inflection

$$x = 0, x = \pm\frac{1}{\sqrt{2}}$$

Note: we didn't plug in values as far as -1 or 1 on the far left and right as that is where the min/max is and didn't want to pick that one up iv.

Concavity:

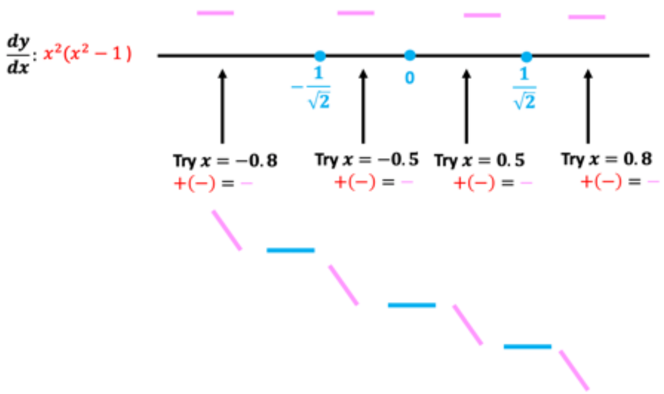
Concave downward when $\frac{d^2y}{dx^2} < 0$

We already have the points where $\frac{d^2y}{dx^2} = 0$

We put these on a number line and check the signs

$+$ means concave down

$-$ means concave up



Concave down when $x \leq -\frac{1}{\sqrt{2}}$, $0 < x < \frac{1}{\sqrt{2}}$